Marianopolis College

Honours project report

Fractals

A study of their generation and their properties

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Abstract

In this project, the study will be on fractals and their properties, mainly their generation via iterated function systems, their dimensions measured by the Hausdorff dimension, and how the nature of the iterated function systems used can also be used to determine the dimension of fractals.

Contents

1 Introduction

1.1 Project summary

It is common knowledge that the dimension of a set can be equal to 0 (for points), 1 (for a line), 2 (for a plane) or 3 (for a space). However, not all sets are as ”well-behaved” as these. This project’s goal is to explore the interesting features of some unusual sets — fractal sets, whose dimension is not an integer. To be precise, the dimension of sets with dimension between 0 and 1 and between 1 and 2 will be examined in more detail. These sets are usually self-similar: if one magnifies a small piece of a fractal set by a large factor, one will obtain an exact copy of the original set. This project aims to study several examples of fractal sets in order to explore some of its interesting properties. Its focus will be on the following topics: metric space and Hausdorff distance, contracting mapping theorems, iterated function systems, sample fractal dimension (for example: Cantor set) and Hausdorff measure and dimensions.

1.2 Fractal sets: an overview

Objects in real life are usually understood as having ”well-defined” dimensions. For example, anything printed on paper, such as pictures, signs and words, are two-dimensional, whereas solids such as tables and chairs are three-dimensional. From a more abstract and mathematical perspective, the ”well-defined”, or Euclidean, dimensions, can be summarized as thus:

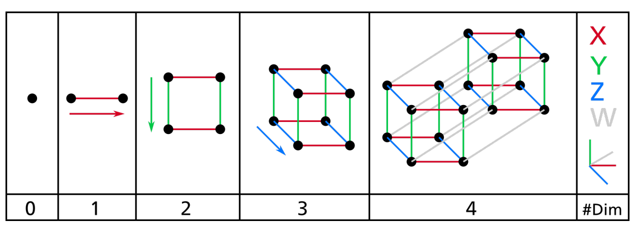


Figure 1: Dimensions 0 to 4 in Euclidean geometry

In general, the dimension of a space is defined as the degree of freedom of the points within it. For example, exactly two variables x, y is needed to define any point $(x, y) \in \mathbb{R}^2$. However, not only do spaces have dimensions, subsets of spaces, such as any point, line, plane or solid $\in \mathbb{R}^3$ can be attributed a dimension too (these objects have dimensions 0 to 3, respectively). But how are the dimension of these objects defined? Can there be objects with non integer dimensions? These questions will be explored in more detail in section 5 (p.13), but it turns out that there do exist objects whose dimensions are non-integer. Consider the construction on figure 2 on page 4

If the illustrated process is repeated infinitely, the end result, called the Koch snowflake, will be a snowflake-like object with no smooth surface, but which exhibits remarkable self-similarity:

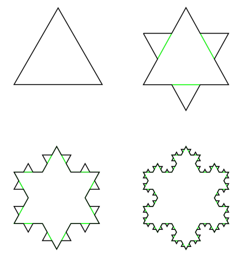


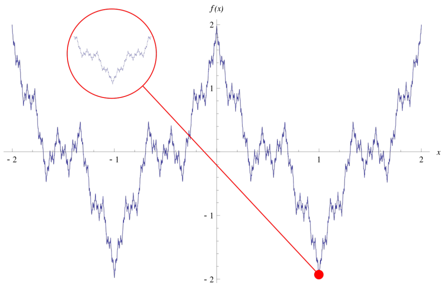
Figure 2: The Koch snowflake

if one keeps zooming into a section of the Koch snowflake, the same patterns will keep repeating. Fractals are often defined (informally) as rough objects that are somewhat self-similar. The Koch snowflake is in many ways hard to analyze using ”classical” mathematics, that is, using calculus and geometry. Calculus only examines functions that are smooth (derivable) at least sometimes, but the Koch snowflake is never derivable. On the other hand, one may note that at step n, the length of a segment (a third) of the Koch snowflake is $l = (\frac{4}{3})^n$. Since $\lim_{n\to\infty} l \to \infty$, the perimeter of the snowflake $\to \infty$. However, since lines have no thickness, the area *occupied* by the perimeter (*not* the area *enclosed* by it) is zero. It is therefore clear that the dimension of this perimeter is higher than two, but less than three. Such objects with non-integer dimensions are beyond the scope of geometry.

Fractals were unknown to mathematicians throughout most of history. The modern understanding of fractals started with Gottfried Leibniz during the 17th century, who wrote about recursive patterns, which are one of the most familiar examples of perfect self-similarity in everyday life. A recursive pattern, or recursion, is an image or pattern that contains itself in its entirety. In the following years, mathematicians considered such patterns to be particularly problematic and little study was done in the area. It was only two centuries later, during the 19th century, that serious breakthroughs were done in the area, with Karl Weierstrass being the first to describe what is now called the Weierstrass function in 1872, which is continuous everywhere, but differentiable nowhere, as shown on figure 3a 5.

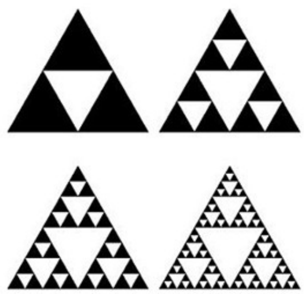
Soon after, throughout the 19th and 20th century, more and more fractal patterns were discovered and described by mathematicians such as Georg Cantor (figure 3b), Helge von Koch (figure 2 at p. 4) and Wacław Sierpiński (figure 3c). These patterns, named in their honour, are some of the most well-known fractals today. Around the same time, Gaston Julia and Pierre Fatou, working independently on complex functions, have both discovered that attractors, the set of values towards which a repeated system of functions tend to evolve (more on iterated function systems in section 4 at p. 12), also exhibit fractal behaviour (figure 3d).

With the increasing interest in these beautiful and intricate patterns, the theoretical aspect of their analysis also saw a breakthrough shortly after, around 1918, when Felix Hausdorff first formulated his new concept of dimension, breaking ground for the concept of non-integer dimensions

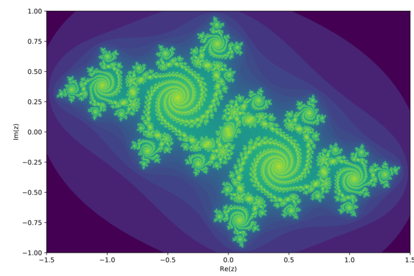


(a) The Weierstrass function

(b) The Cantor set



(c) The Sierpiński triangle



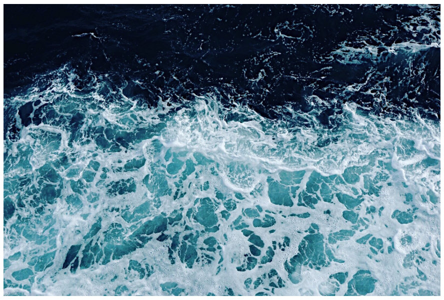
(d) The Julia set

Figure 3: Examples of fractals

and essentially founding the theoretical framework of modern fractal analysis (more on his works in section 2 at p. 6 and section 6 at p. 16).

It may seem that fractals are of little use beyond their artistic value and being an interesting area of research for theoretical mathematicians, it would turn out that many natural phenomena can be described as being fractals. In fact, objects in nature have the tendency of being rough and that roughness has the tendency to stay consistent if one zooms in. Figure 4 on page 6 contains some real life examples that are best described as fractals, where they exhibit either some roughness that essentially never disappears as one zooms in (4a and 4b), or some extensive branching that (4c and 4d).

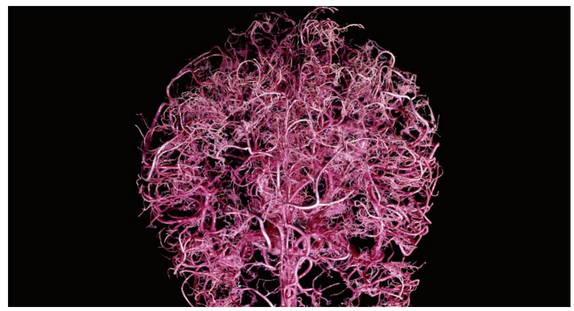
Due to the prevalence of fractal patterns in nature, it would be incredibly useful to study the ways of generating new fractals to model these phenomena. The following sections will study and explain in detail the generation of fractals from iterated function systems, which is used and studied extensively, among others, in computer science as a simple and straightforward way for computers to generate fractals.



(a) Surface of turbulent water

(b) Frost

(c) Growth pattern of trees



(d) Human blood vessels

Figure 4: Real life examples of fractals

2 Metric spaces and Hausdorff distance

In order to understand iterated function systems, many mathematical concepts essential to the study of fractals need to be introduced. People familiar with geometry can easily understand the concept of Euclidean space, but to study sets and the relationships between themselves during a step-wise generation of a fractal, a new, more general, concept of space is needed.

2.1 Metric space

From linear algebra, it is known that the distance function $d(x, y)$ in Euclidean space is defined in $\mathbb{R}^n$ as:

$$d(x, y) = \sqrt{\sum_{i=1}^{n}(y_i - x_i)^2}; x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n \tag{1}$$

Of course, there is much more to Euclidean spaces, but the fact that equation 1 is defined is enough to qualify Euclidean spaces as metric spaces. More formally, a set M is called a *metric space* if there exists a function $d(x, y) \in \mathbb{R}$ such that

Definition 2.1. A set M, with elements $x, y, z \in M$ is called a **metric space** if there exists a function $d(x, y) \in \mathbb{R}$ such that the following hold $\forall x, y, z$.

1. $d(x, y) = 0 \Leftrightarrow x = y$ (identity of indiscernibles)

2. $d(x, y) = d(y, x)$ (symmetry)

3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangular inequality)

If these 3 conditions are all respected, one can easily derive that $d(x, y) \geq 0$ (positivity), as follows:

Proof.

$$d(x, x) \leq d(x, y) + d(y, x)$$
$$\because d(x, x) = 0 \text{ and } d(x, y) = d(y, x)$$
$$\therefore 0 \leq d(x, y) + d(x, y)$$
$$\therefore 0 \leq d(x, y)$$

□

Although the conditions are all somewhat intuitive with regards to their application to Euclidean spaces, one can still verify that they hold for equation 1 ($a, b, c \in \mathbb{R}; x, y, z \in \mathbb{R}^n$):

Proof. $\because a^2 = 0 \Leftrightarrow a = 0$ and $a - b = 0 \Leftrightarrow a = b$
$\therefore d(x, y) = 0 \Leftrightarrow x_i = y_i; i = 1, 2, \ldots, n \Leftrightarrow x = y$ □

Proof. $\because (a - b)^2 = (b - a)^2$
$\therefore d(x, y) = d(y, x)$ □

Proof. Let $\vec{u} = [y_i - x_i]$ and $\vec{v} = [z_i - y_i]$, such that $\left\|\vec{u}\right\| = d(x, y), \left\|\vec{v}\right\| = d(y, z)$ and $\left\|\vec{u} + \vec{v}\right\| = \|[z_i - x_i]\| = d(x, z)$.
The triangular inequality for vector spaces $\left\|\vec{u} + \vec{v}\right\| \leq \left\|\vec{u}\right\| + \left\|\vec{v}\right\|$ can be shown using the Cauchy-Schwarz inequality and will be proven in Annex I. $\therefore d(x, z) \leq d(x, y) + d(y, z)$ □

What these conditions accomplish is a generalization of the concept of distance from Euclidean spaces. Many theorems from Euclidean spaces can be generalized into all metric spaces. However, in order to utilize the interesting properties of metric spaces on sets, one must first define a $d(x, y)$ for sets in order to turn the set A of all subsets of $\mathbb{R}^n$ into a metric space itself. In other words, a definition of distance between sets is needed in order to progress.

2.2 Hausdorff distance

Although the concept of defining a distance function between sets sounds daunting, the basic idea behind it is quite intuitive: the more similar two sets are, both in space and in "shape", the small the distance between them is. One starts by defining the distance between a point x and a set Y, both $\in \mathbb{R}$, as

$$d(x, Y) = \inf_{y \in Y} d(x, y) \tag{2}$$

or, informally, as the smallest possible $d(x, y); y \in Y$. It is clear that this function does not respect the identity of indiscernibles since $d(Y, Y)$ is not defined. For symmetry, similarly, $d(Y, x)$ is not defined either. Nonetheless, using this equation as a starting point, one can expand it to define the distance between two sets $X, Y \in \mathbb{R}^n$ (here, it is *different* from the Hausdorff distance, although elsewhere "distance between two sets" and "Hausdorff distance" may be used interchangeably) as the longest possible $d(x, Y); x \in X$.

$$d(X, Y) = \sup_{x \in X} d(x, Y) = \sup_{x \in X} \inf_{y \in Y} d(x, y) \tag{3}$$

This respects the identity of indescernibles, since $d(x, X)$ is always 0. However, it does not respect symmetry, since $d(X, Y)$ is clearly defined differently from $d(Y, X)$. To rectify this, $d_H(X, Y)$, the Hausdorff distance, is simply defined as the larger value between $d(X, Y)$ and $d(Y, X)$.

Definition 2.2. For sets $X, Y \in \mathbb{R}^n$, the **Hausdorff distance** $d_H(X, Y)$ is defined as thus:

$$\begin{aligned} d_H(X, Y) &= \max\{d(X, Y), d(Y, X)\} \\ &= \max\left\{\sup_{x \in X} d(x, Y), \sup_{y \in Y} d(X, y)\right\} \\ &= \max\left\{\sup_{x \in X} \inf_{y \in Y} d(x, y), \sup_{y \in Y} \inf_{x \in X} d(x, y)\right\} \end{aligned} \tag{4}$$

This, again, respects the identity of indescernibles, but also that of symmetry, since $d_H(X, Y)$ is defined symmetrically. On the other hand, to show that it respects the triangular inequality rigorously, although crucial to any further study into this method of generating fractals, is beyond the scope of this report. Here, only a sketch of the proof will be given to help one's intuition on the validity of d_H as a distance metric.

Proof. Sketch of proof that $d_H(X, Y)$ respects the triangular inequality

1. Let X, Y, Z be sets $\in \mathbb{R}^n$.

2. From the triangular inequality for points, if there is a way for $d_H(X,Z) > d_H(X,Y) + d_H(Y,Z)$ (the counter-example) to hold, one may notice that there must be a way, without modifying $d_H(X,Z)$, to translate and rotate Y such that the three d_H are aligned, while $d(X,Y') \leq d(X,Y)$ and $d(Y',Z) \leq d(Y,Z)$.
   The result of that would be $d_H(X,Z) > d_H(X,Y) + d_H(Y,Z) \geq d_H(X,Y') + d_H(Y',Z)$. In other words, the triangular inequality must not hold in $\mathbb{R}^1$ for the initial counter-example to exist.

3. Therefore, if one proves that the triangular inequality holds for d_H in $\mathbb{R}$, it must also hold in $\mathbb{R}^n$.

Lemma 2.1. *Let* $X = (x_1, x_2)$*,* $Y = (y_1, y_2)$ *and* $Z = (z_1, z_2)$ *be sets* $\in \mathbb{R}$ *(it makes no difference whether any of the extremities are excluded or not).*
$d_H(X,Z) \leq d_H(X,Y) + d_H(Y,Z)$

Remark. Observe that $d_H(X,Y) = \max\{d(x_1,y_1), d(x_2,y_2)\}$
(same principle for $d_H(X,Z)$ and $d_H(Y,Z)$).

Proof. Proof that d_H holds for $\mathbb{R}$:

1. If $d_H(X,Y) = d(x_1,y_1)$, $d_H(Y,Z) = d(y_1,z_1)$ and $d_H(X,Z) = d(x_1,z_1)$, it is clear from the triangular inequality for $\mathbb{R}$ that it also holds for d_H;

2. If $d(x_2,z_2) \geq d(x_1,z_1)$, the rest staying unchanged, then
   $d_H(X,Z) = d(x_2,z_2) \leq d(x_2,y_2) + d(y_2,z_2) \leq d(x_1,y_1) + d(y_1,z_1) = d_H(X,Y) + d_H(Y,Z)$
   which is the triangular inequality.

3. If $d(x_2,y_2) \geq d(x_1,y_1)$, then
   $d_H(X,Z) = d(x_1,z_1) \leq d(x_1,y_1) + d_H(y_1,z_1) \leq d(x_2,y_2) + d(y_1,z_1) = d_H(X,Y) + d_H(Y,Z)$
   which is the triangular inequality (same principle for if $d(y_2,z_2) \geq d(y_1,z_1)$).

4. Swapping all subscripts (replacing x_1 with x_2 etc) from above step would account for all possible situations in $\mathbb{R}$. Therefore, d_H holds in $\mathbb{R}$.

□

Since a counter-example cannot exist in R, it is impossible for it to exist in R^n. Therefore, $d_H(X,Z) \leq d_H(X,Y) + d_H(Y,Z)$. □

2.3 Applications

Besides allowing further study of the generation of fractals, the concept of distance between sets also allows a general quantification of similarity between sets, while quantification is very important in computer science. In fact, Hausdorff distances are used widely in graphics recognition to evaluate how similar an arbitrary image is to a template. Now that it is established that there exists a distance between sets, the next section will deal with a seemingly unrelated subject: contraction transformations and their properties.

3 Contraction mapping theorem

If one analyzes the transformation applied at each step for the Cantor set in figure 3b back at page 5, one may easily notice that the middle third of each section is removed at each step. However, an alternative way to describe the steps could be that the set at each step is *shrank* by $\frac{1}{3}$ on two instances, with the second one then translated to the right by $\frac{2}{3}$. To study contractions could therefore produce insight on the nature and properties of the transformation that is being applied to fractal sets at each step.

3.1 Contraction mapping

A contraction, quite intuitively, is defined as thus:

Definition 3.1. Let T be a transformation and V be a subset of a metric space such that $T : V \to V$. Let x, y be arbitrary elements of V. T is called a **contraction** if

$$d(T(x), T(y)) \leq cd(x, y) \tag{5}$$

where $0 < c < 1$ is a constant.

Informally, a contraction is a transformation of a set onto itself where the distance d has shrank by at least a constant. Taking the example of the Cantor set, there are two contractions T_1 and T_2 that are constant throughout each step, for which $c = \frac{1}{3}$ and $d(T(x), T(y)) = cd(x, y) = \frac{1}{3}d(x, y)$.

Although this transformation does not seem like much, it is the origin of many important topological properties. One of them, the contraction mapping theorem, or the Banach fixed-point theorem, will be of particular importance to fractals and will be discussed in detail. Before that, however, a few restrictions must be placed on the set V. Namely, no point must be missing from V.

3.2 Cauchy sequences and completeness

The concept of Cauchy sequences is very close to that of a converging sequence. In fact, a sequence is converging if and only if it is also Cauchy. Let $\{a_i\} \in \mathbb{R}$ be a sequence. It is defined as Cauchy if $|a_m - a_n|\,; m > n$ gets arbitrarily small as $n \to \infty$. In other words, past a finite n, the distance between any two terms get arbitrarily small. The generalization of the concept of distance onto any metric space M, instead of just on $\mathbb{R}$, would yield the generalized definition of a Cauchy sequence.

Definition 3.2. Let the sequence $\{x_i\}$ be elements of metric space M. $\{x_i\}$ is called a **Cauchy sequence** if, $\forall \epsilon > 0$, $\exists$ some N such that $d(x_n, x_m) < \epsilon \forall n, m > N$.

Besides being equivalent to a sequence being converging, Cauchy sequences also have important applications in analysis. Here, it can be used to measure whether a set is missing any points. If a set is missing points, then there must be some Cauchy sequences from the elements of the set that converge to the missing points. Therefore, there is reason to claim that, if a set V contains the limits of all Cauchy sequences formable from its elements, that set can be called complete.

Definition 3.3. Let the set V be metric. Let $\{x_i\}$ be an arbitrary Cauchy series of elements $\in V$. V is **complete** if

$$\lim_{i\to\infty} x_i \in V \tag{6}$$

3.3 The theorem

Theorem 3.1 (Contraction mapping theorem). *Let the set V be metric. Let $T : V \to V$ be a contraction. There exists one and only one fixed point $x \in V$ such that $T(x) = x$.*

Simply put, this theorem points out that as long as the set is not missing any points (i.e. is complete), after applying any given contraction that maps the set onto itself (i.e. no points will, after the transformation, "move" out of the set), it is always true that every single point in the set except one (i.e. the fixed point) will have "moved" some distance. In fact, if one thinks about scaling pictures in Photoshop, this makes perfect sense since every point will have moved towards some point on the picture while that point stays fixed.

A complete proof of this theorem will be given in Annex I (p. 21). Here, a general idea of the proof would be given. First, examining $d(T^{(n)}(x), T^{(n+1)}(x))$ for any arbitrary $x \in V$, one will notice that not only does that distance tend towards 0 as $n \to \infty$, but $d(T^{(n)}(x), T^{(m)})$ tends towards 0 as well as $n, m \to \infty$, due to properties of geometric series. This means that $T^{(i)}(x)$ is a Cauchy sequence, meaning that by the definition of a complete set its limit L must $\in V$. That limit would have the property of staying fixed, since $\lim_{n\to\infty} T^{(n+1)}(x) = \lim_{n\to\infty} T^{(n)}(x) = L$. L would therefore be a fixed point. Second, there can't be two or more fixed points, since the distance between them would stay fixed after each transformation, which violates the definition of a contraction.

3.4 Scope and applications

Thinking in terms of fixed points and the existence of them is helpful in many areas of mathematical research. For the contraction mapping theorem, since it can be generalized to metric spaces, it can be used to prove the existence of objects such as solutions to differential equations as long as the set of functions can be mapped as some sort of metric space while some sort of contraction is being applied to them. Also, contraction mapping theorem has the more obvious application in geometry and topology for fixed points in Euclidean space. Finally, theorems like this that are generalized to all metric spaces, beyond Euclidean spaces, make the concept of metric space that much useful.

4 Iterated function systems

Now that the proper mathematical background has been presented, they can finally be put all together to help study one very useful way of generating fractals: that of iterated function systems. As seen before, the fractals seen so far can all be understood as being generated via contractions. Put simply, an iterated function system is the set of all contractions applied at each step during the generation of a fractal.

4.1 Definition

Definition 4.1. Let $T_1, T_2, \dots, T_N$ be a finite set of contractions that map complete metric set V onto itself. Let x be an arbitrary element $\in V$. The **iterated function system** $F : V \to V$ is defined as

$$F(x) = T_1(x) \cup T_2(x) \cup ... \cup T_N(x) \tag{7}$$

In the case of $T_i(x)$, they are simply contractions acting on Euclidean space for the fractals mentioned in this report. Of course, since $F(x)$ is expected to be a fractal-generating transformation, it would only be efficient if there exists a unique fractal for all $F(x)$. This turns out to be true, since $F(x)$, besides being made of contractions in the more intuitive sense, are themselves contractions acting on sets. To understand this, recall that distance between sets is a measure of how similar they are. Since $F(x)$ is what is happening to the set at each step during the generation of a fractal, by examine the step-wise generations on figure 2 (p. 4) or 3 (p. 5), one can remark that the sets are changing less and less at each step, meaning that they are getting more and more similar. Therefore, it is plausible that IFS is in fact a contraction, not on the Euclidean space, but on the space of all subsets in Euclidean space (which, via the Hausdorff distance, is also a metric space).

4.2 Attractors

To prove that F is a contraction is beyond this report. However, some general ideas on the proof can be given. Since the $d_H(X, Y)$ is always based on the distance between some points x and y from their respective sets (section 2 at p. 6) and *all* distances are scaled down by the set of contractions, $d_H(X, Y)$ must also scale down after each step, thus showing that F is a contraction. Then, by the contraction mapping theorem, there must be some set $\in \mathbb{R}^n$ (since F is a contraction acting on the *space containing all subsets* $\in \mathbb{R}^n$) that is fixed after being transformed by F.

Definition 4.2. A set $S \in \mathbb{R}^n$ is called an **attractor** if, for some iterated function system F,

$$F(S) = S \tag{8}$$

The contraction mapping theorem guarantees both the existence and the uniqueness of the attractor for a given F. It happens that attractors are fractals in general, thus making the generation of fractals through iterated function systems an *attractive* method to obtain new fractals. As concrete examples, the iterated function system that generates the Cantor set (p. 5) and the Sierpiński triangle (p. 5) will be examined:

1. The middle third Cantor set: since each segment is scaled down by $\frac{1}{3}$, one needs simply to observe that $T_2(x)$ is translated by $\frac{2}{3}. Thus, T_1(x) = x/3$, and $T_2(x) = (x + 2)/3$; $x \in [0, 1]$.

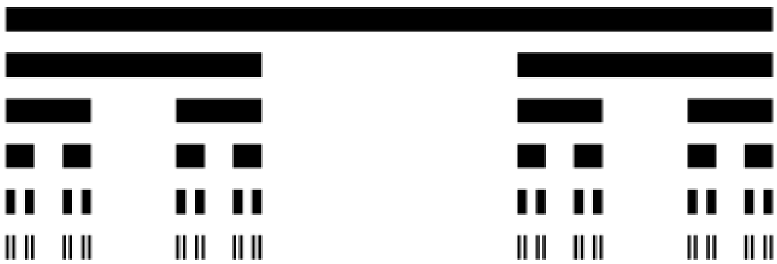
2. Sierpiński triangle: since each segment is scaled down by $\frac{1}{2}$, one needs simply to observe the translation on T_2 and T_3 compared to T_1. $T_1(x, y) = (x/2, y/2)$; $T_2(x, y) = ((x + 1)/2, y/2)$; $T_3(x, y) = (x/2 + 1/4, y/2 + \sqrt{3}/4)$.

4.3 Applications

A way to generate a wide variety of fractals that is easy for computers to generate has many interesting applications, many of which have been discussed either in the introduction, or here and there along the report. One of the most interesting applications, still, is that many things in nature can be made easier to model, especially when the object of interest has either a lot of branching (concerns recursion, which is one of the fundamental fractal concepts), or has a lot of roughness (which can be modelled via some method of generating fractals, for example via iterated function systems).

5 Fractal dimensions

A detailed calculation for Cantor set is as follows:



Length:

The length of the Cantor set will be 0.

The length of the remaining intervals after step1 is $2/3$;

The length is $(2/3)^2$ after step 2;

The length is $(2/3)^n \rightarrow 0$ after step n.

Dimension:

Analogy with traditional geometry:

For 1D, assume we need N(a)intervals to cover the set I=[0,1]. Suppose $a = 1/n$, then $N(a) = a^{-1} = n$;

For 2D, we need N(a)small squares to cover the set $I = [0, 1]^2$. Suppose $a = 1/n$, then $N(a) = a^{-2} = n^2$;

Deduce from the above, the rest of the sets would follow the following pattern:

To cover set $I = [0, 1]^D$, we need$N(a) = a^{-D} = n^D$ elements.

To cover the Cantor set:

$\because$ After step1,scale C by a factor $a = 1/3$, we need2intervals to cover C

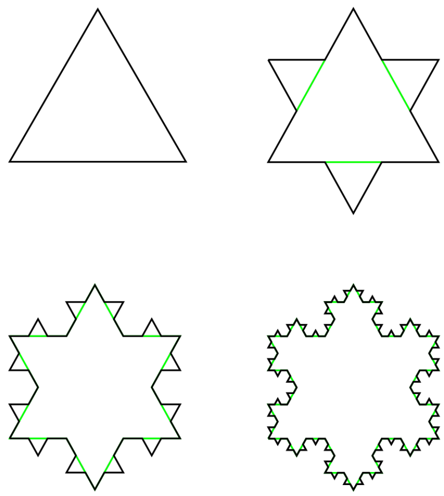
$\therefore N(a) = a^{-D} = 1/3^{-D} = 2 \quad \rightarrow 3^D = 2$

$\therefore D = log_3 2$

To double check:

After step2, $N(a) = (1/3)^2 = 4 = 2^2$, therefore, D is still $log_3 2$.

A detailed calculation of Von Koch snowflake's area and dimension:



Area:

Assume the area of the original triangle is A_0.

After step1, the added number of triangle $= 3 = 3 \times 4^{1-1}$

After step2, the added number of triangle $= 12 = 3 \times 4^{2-1}$

Deduce from above, after step n,the added number of triangle $= 3 \times 4^{n-1}$

Moreover,after step1, the added area of individual triangle $= A_0/9$

After step2, the added area of individual triangle $= A_1/9 = A_0/9^2$

Deduce from above, after step n,the added area of individual triangle $= A_{n-1}/9 = A_0/9^n$

Therefore, the total area after step n will be:

$$A_0 + \sum_{i=1}^{\infty} 3 \times 4^{i-1} \times A_0/9^n = A_0 \times (1 + \sum_{i=1}^{\infty} 3/4 \times (4/9)^i) = A_0 \times (1 + 3/4 \times \sum_{i=1}^{\infty} (4/9)^i) = 8/5 \times A_0 \quad (9)$$

Dimension

For Koch snowflake, its set is similar to Cantor set, thus the calculation is similar.

Focus on one side of K; other sides follow the same process

$\because$ After step1,scale K by a factor $a = 1/3$, we need 4 intervals to cover K

$\therefore N(a) = a^{-D} = 1/3^{-D} = 4 \quad \rightarrow 3^D = 4$

$D = log_3 4 = 2 log_3 2$

The result is reasonable since the snowflake has infinite length$\rightarrow$ it has a dimension greater than 1

It is not enough to cover the plane$\rightarrow$ it has a dimension smaller that 2.

6 Hausdorff measure and dimension

Hausdorff measure is used to determine Hausdorff dimension, where the latter has the advantage of being defined for any set, especially for irregular fractal sets, which makes it essential for the understanding of fractals.

6.1 Hausdorff measure

Hausdorff measure assigns a number in [0,∞]to each set in $\mathbb{R}^n$, or more generally, in any metric space. The zero-dimensional Hausdorff measure is the number of points in the set if the set is finite, or ∞if the set is infinite. The one-dimensional Hausdorff measure is the length of the line/curve in the set if the set is finite, or ∞if the set is infinite. Deduce the rest from the above, Hausdorff measure will generalize counting, length, area , volume and so on.

For a simple example, let F be a flat disc of unit radius in R^3. From properties of length, area and volume, H^1(F) =length (F) = ∞ , $0 < H^2(F)$ =$(4/\pi)\times$ area(F)= $4 < \infty$, and $H^3(F) = (6/\pi) \times$ vol(F) = 0. Therefore, the disc lies in dimension 2 since its Hausdorff measure gives an finite result that is between 0 and ∞.

In a n-dimensional Euclidean space $\mathbb{R}^n$, the diameter of U is defined as

$$|U| = \sup\{|x - y| \ : x, y \in \ U\} \tag{10}$$

which is the greatest distance apart of any pair of points in U. If U_i is a countable collection of sets of diameter at most δ that cover F, we say that U_i is a δ -cover of F.

Suppose F is a subset of $\mathbb{R}^n$ and $s$is a non-negative number. For any δ ¿ 0, we have the proposition:

$$H^s{}_\delta(F) = \inf\{\sum_{i=1}^{\infty}|U_i|^s : \{U_i\}\text{is a } \delta\text{-cover of F}\} \tag{11}$$

The objective of δ-cover is to minimize the sum of the sth powers of the diameter.

The above graphs show the two possible δ-cover of set F, the infimum would be the one whose sth powers has less diameter. The infimum of $\sum|U_i|^s$ over all such δ-covers $\{U_i\}$ gives, $H^s_\delta(F)$.

For any fixed cover U_k of F, such that diameter of, $U_k < 1$, for any number k $\sum|U_k|^s$ is a decreasing function of s. $H^s(F)$, called s-dimensional Hausdorff measure of F, is a non-increasing function of s.We thus have the following proposition:

$$\text{Suppose } H^s(F) < +\infty, \text{then } H^t(F) = 0 \quad \text{for t>s} \qquad (6.1) \tag{12}$$

6.2 Hausdorff Dimension

Hausdorff dimension, also known as fractal dimension, is a measure of chaos that was first introduced by mathematician Felix Hausdorff in 1918.

The Hausdorff dimension of a single point is 0, of a line segment is 1, of a square is 2, and of a cube is 3. That is, for a traditional geometry, the Hausdorff dimension is an integer agreeing with the usual sense of dimension. However, fractals have non-integer Hausdorff dimensions as seen in the calculations in section 1.2.

The definition of Hausdorff dimension is based on Hausdorff measure:

$$dim_H F = \inf\{s \geq 0 : H^s(F) = 0\} = \sup\{s : H^s(F) = \infty\} \tag{13}$$

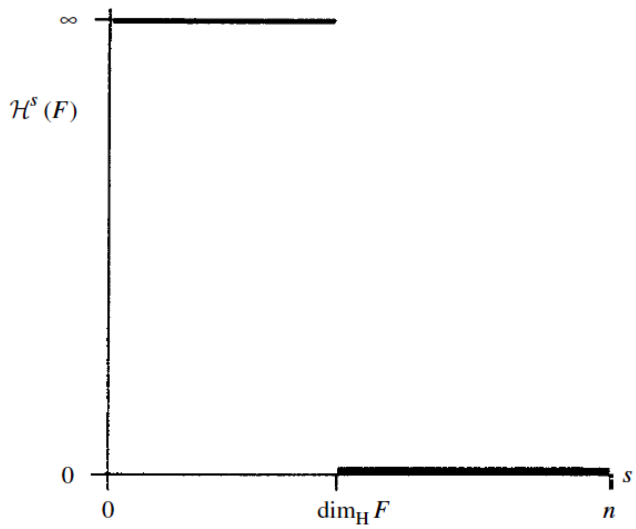
That is to say, Hausdorff measure and Hausdorff dimension satisfies the following relation:

$$H^s(F) = \begin{cases} \infty, & \text{if } 0 \leq s < dim_H F \\ 0, & \text{if } s > dim_H F \end{cases} \tag{14}$$

If $s = dim_H F$, then $H^s(F)$ may be zero or infinite, or may satisfy

$$0 < H^s(F) < \infty \tag{15}$$

As seen in the following graph,Hausdorff dimension of the set is the point s"jumps"from 0to∞.



An example for the application of this graph is similar to the one given in section 6.1: let Fbe a flat disc whose radius is 1. From properties of length, area and volume, $H^1(F)$=length (F)=∞, 0< $H^2(F)$=$(4/\pi)\times$area(F)=4< ∞, and $H^3(F)$= $(6/\pi)\times$vol(F)=0. Thus $dim_H(F)$=2 with $H^s(F)=\infty$ if $s < 2$, and $H^s(F)$=0 if s >2.

6.2.1 Calculation of Hausdorff dimension

For example, as we all know middle-third Cantor set has a dimension $s = log_3 2$ as proven in section 5. Let F=C, then $0 \leq 1/2 \leq H^s(C) \leq 1 \leq \infty$. Therefore, $dim_H(C) = s$.

The proof is as the following:

***Part*1**, prove $H^s(C) \leq 1$

Let $U_k{}^j$ $(1 \leq k \leq 2^j)$ be the cover of Cantor set by 2^j intervals of length $1/3^j$ at step j.

By definition, $H^s{}_\delta(F) = \inf\{\sum_{i=1}^{\infty}|U_i|^s : \{U_i\}$

$\sum_{k=1}^{\infty}\left|U_k{}^j\right|^s = (3^{-j})^s \times 2^j$=$(2\times 3^{-s})^j$=$(2\times \frac{1}{2})^j$=1 $\quad (\delta_j$=$3^{-j})$

***Part*2**, prove $H^s(C) \geq \frac{1}{1}$

Let $U_i$be an interval in a cover of C.

Let $\frac{1}{3^{k+1}} \leq |U_i| \leq \frac{1}{3^k}$ (k$\geq$ 1)

U_i can intersect at most 1 interval of level k in the construction of C.

If $j \geq k$, then $\left|U_i 3\right|^k$ can intersect at most 2^{j-k} intervals of level j.
2^{j-k}=$2^j \times 3^{-sk} \leq 2^j \times 3^s \times |U_i|^s \quad 3^s = \frac{1}{2}$
Choose j so large that $3^{-(j+1)} \leq |U_i|$ for all i.
The intervals U_i have to intersect all 2^j "basic" intervals of level j in the constructions of C.
Therefore, $2^j \leq \sum$(of intervals of level j that U_i intersects) $\leq \sum_i 2^j 3^s |U_i|^s$
$\rightarrow \sum_i |U_i|^s \geq 3^{-s} = \frac{1}{2}$

6.2.2 Hausdorff dimension properties

Hausdorff dimension satisfies the following properties (which might be well hold for any reasonable definition of dimension)

1. Monotonicity. If E ⊂ F, then $dim_H E \leq dim_H F$. This is immediate from the measure property that $H^s(E) \leq H^s(F)$ for each s.

2. Countable stability. If $F_1, F_2, ...$ is a (countable) sequence of sets then $dim_H \cup_{i=1}^{\infty} F_i = \sup_{1 \leq i < \infty}$ $dim_H F_i$. This is to say, $dim_H \cup_{i=1}^{\infty} F_i \geq dim_H F_j$ for each j from the monotonicity property. On th other hand, if s> $dim_H F_i$ for all i, then $H^s(F_i) = 0$, so that $H^s(\cup_{i=1}^{\infty} F_i) = 0$, giving the opposite inequality.

3. Countable sets. If F is countable, then $dim_H F = 0$. Since if F_i is a single point, $H^0(F_i) = 1$ and $dim_H F_i = 0$, thus by countable stability $dim_H \cup_{i=1}^{\infty} F_i = 0$.

4. Open sets. If $F \subset R^n$ is open, then $dim_h F = n$. For because F contains a ball of positive n-dimensional volume, $dim_H F \geq n$,but since F is contained in countably many balls, $dim_H F \leq n$using countable stability and monotonicty properties.

7 Conclusion

In conclusion, the project explored the many proposed topics, including metric space, Hausdorff distance, contraction mapping, iterated function systems, fractal dimension calculations and Hausdorff measure and dimension, while providing the definition of some essential topics among these. The project also examined some simple, perfectly self-similar fractal sets in more detail, such as the Cantor set and the Koch snowflake. In general, this project provided an introduction to various topics related to fractals, which has become an increasingly active area of research as its usefulness is being made more and more apparent partly thanks to computer science. It is also a good starting point for some basic understanding of the related fields such as topology, set theory and analysis.

Appendix I: Proofs

I.I: Proof of triangular inequality in $\mathbb{R}^n$ (for vectors)

As said in the text, this proof entails the Cauchy-Schwarz inequality, which deals with inherent properties of the dot product and does not need the triangular inequality to be proven.

Theorem 7.1 (Cauchy-Schwarz inequality)**.** *Let* $\vec{u}, \vec{v} \in \mathbb{R}^n$

$$\vec{u} \cdot \vec{v} \leq \vec{u} \cdot \vec{u} \tag{16}$$

Proof. Proof of the Cauchy-Schwarz inequality.

First, there is clear an equality if the norm of either vector is 0. Therefore, for the rest of the proof, it can assumed that $\|\vec{v}\|, \|\vec{u}\| \neq 0$. Let $\vec{t} \in \mathbb{R}^n$ be such that

$$\vec{t} = \frac{\vec{u}}{\|\vec{u}\|} - \frac{\vec{v}}{\|\vec{v}\|} \tag{17}$$

$$\begin{aligned}
\vec{t} \cdot \vec{t} &= \left(\frac{\vec{u}}{\|\vec{u}\|} - \frac{\vec{v}}{\|\vec{v}\|}\right) \cdot \left(\frac{\vec{u}}{\|\vec{u}\|} - \frac{\vec{v}}{\|\vec{v}\|}\right) \\
&= \frac{\vec{u} \cdot \vec{u}}{\|\vec{u}\|^2} - 2\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} + \frac{\vec{v} \cdot \vec{v}}{\|\vec{v}\|^2} \\
&= 2 - 2\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|}
\end{aligned}$$

$$\begin{gathered}
\because \vec{t} \cdot \vec{t} = \|\vec{t}\|^2 \geq 0, \\
2 - 2\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} \geq 0 \\
2 \geq 2\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} \\
\|\vec{u}\|\|\vec{v}\| \geq \vec{u} \cdot \vec{v}
\end{gathered}$$

□

Proof. Proof of triangular inequality in $\mathbb{R}^n$ (for vectors)

$$
\begin{aligned}
\left\|\vec{u}+\vec{v}\right\|^2 &= (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) \\
&= \vec{u}\cdot\vec{u} + 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v} \\
&= \left\|\vec{u}\right\|^2 + 2\vec{u}\cdot\vec{v} + \left\|\vec{v}\right\|^2, \because \left\|\vec{u}\right\|\left\|\vec{v}\right\| \geq \vec{u}\cdot\vec{v} \\
&\leq \left\|\vec{u}\right\|^2 + 2\left\|\vec{u}\right\|\cdot\left\|\vec{v}\right\| + \left\|\vec{v}\right\|^2 \\
&= (\left\|\vec{u}\right\| + \left\|\vec{u}\right\|)^2
\end{aligned}
\tag{18}
$$

$$\because \text{ all norms} \geq 0, \left\|\vec{u}+\vec{v}\right\| \leq \left\|\vec{u}\right\| + \left\|\vec{u}\right\|$$

□

I.II: Proof of the contraction mapping theorem

Proof. This proof can be done in two parts, the first of which proves the existence of fixed points, while the second proves that only one fixed point exists.

Proof of existence of a fixed point: Let x be any arbitrary element $\in$ complete metric set V. Let $T : V \to V$ be a contraction by at least a factor of $0 < c < 1$.

$$
\begin{gathered}
\because x, T(x) \in V, \therefore \text{ by definition of a contraction,} \\
cd(x, T(x)) \geq d(T(x), T(T(x))) = d(T(x), T^{(2)}(x)) \\
\therefore c^2 d(x, T(x)) \geq cd(T(x), T^{(2)}(x)) \geq d(T^{(2)}(x), T^{(3)}(x))
\end{gathered}
$$

Applying T n-times, we obtain:

$$c^n d(x, T(x)) \geq d(T^{(n)}(x), T^{(n+1)}(x))$$

Let constant $a = d(x, T(x))$.

$$c^n d(x, T(x)) = c^n a \geq d(T^{(n)}(x), T^{(n+1)}(x))$$

Because of the triangular inequality for $d(x_1, x_2)$, for any $m > n$,

$$
\begin{aligned}
d(T^{(n)}(x), T^{(m)}(x)) &\leq d(T^{(n)}(x), T^{(n+1)}(x)) + \cdots + d(T^{(m-1)}(x), T^{(m)}(x)) \\
&\leq \sum_{i=n}^{m-1} d(T^{(i)}(x), T^{(i+1)}(x)) \quad \leq \sum_{i=n}^{m-1} c^i a \\
&= a\frac{(c^n - c^m)}{c-1} \text{ by the sum of geometric series.}
\end{aligned}
$$

Let constant $b = \frac{a}{c-1}$.

$$
\begin{gathered}
b(c^n - c^m) = bc^n(1 - c^{m-n}) \\
\because 0 < c < 1 \text{ and } m > n, \\
\therefore \lim_{n\to\infty} bc^n(1 - c^{m-n}) = b \times 0(1 - 0) = 0,
\end{gathered}
$$

$\therefore\ d(T^{(n)}(x), T^{(m)}(x)) \leq bc^n(1 - c^{m-n}) < \epsilon$ for any ϵ arbitrarily close to 0 if one takes an n large enough. Therefore, by the definition of Cauchy sequences, $\{T^i(x)\}$ is a Cauchy sequence.
Let L be the limit of $\{T^i(x)\}$ as $i \to \infty$. By the definition of a complete set, $L \in V$. If one applies T to L,

$$T(L) = T(\lim_{i\to\infty} T^{(i)}(x)) = \lim_{i\to\infty} T^{(i+1)}(x) = L$$

which is the definition of a fixed point.

Proof of uniqueness of the fixed point: suppose there exists two fixed points x, y in the complete metric space V. Let $T : V \to V$ be a contraction by at least a factor of $0 < c < 1$. Let By definition of fixed points, $T(x) = x$ and $T(y) = y$.

$d(T(x), T(y)) = d(x, y) > cd(x, y)$ which is a contradiction against the definition of a contraction.

Therefore, there cannot exist two fixed points, which proves the uniqueness of the fixed point. □

I.III: Proof of the proposition 6.1

Proof: Suppose $H^s(F)$= A< ∞

Then $\ni \delta_j \to 0$ and ${U_k}^j$ such that diameter $({U_k}^j) < \delta_j$ for all k, and $\sum_{k=1}^{n_j} \left|{U_k}^j\right|^s < A + 1$

Set $t < s$

$\sum_{k=1}^{n_j} \left|{U_k}^j\right|^t = \sum_{k=1}^{n_j} \left|{U_k}^j\right|^s \times \left|{U_k}^j\right|^{t-s} < \delta_j^{t-s} \sum_{k=1}^{n_j} \left|{U_k}^j\right|^s < \delta_j^{t-s}(A + 1)$

Take $lim_{\delta_j} \to 0$

Since $H^t_\delta(F) = \inf\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\}$is a δ-cover of F, then we have $H^t(F) = 0$ for any $t > s$

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