Note on the spectra of some sequences of regular graphs

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Abstract

Kollár and Sarnak [4] have investigated the spectral set of sequences of 3-regular graphs, using in particular the map from a graph to the line graph of its subdivision (so-called vertex replacement map in this note). We generalize this map and some results of the paper to k-regular graphs.

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1 The vertex replacement map

Let $\mathcal{F}_k$ be the set of k-regular (simple, undirected) graphs ($k \geq 2$ across this section). For a given k we will define a map from $\mathcal{F}_k$ onto itself that will help us generate k-regular graphs, via vertex replacement. This map has been investigated by Kollár and Sarnak in [3] and [4] for 3-regular graphs.

Definition 1.1. Let G be a graph. $D(G) = D_G$ (called the *subdivision graph* of G) is defined to be the graph created from subdividing every edge in $E(G)$. Formally, $V(D_G) = E(G) \cup V(G)$ and $E(D_G) = \{(e, v) : e \in E(G), v \in V(G), e \text{ incident to } v\}$.

Remark. For $x \in V(D_G)$, if $x \in V(G)$ then $\deg_{D_G}(x) = \deg_G(x)$ and if $x \in E(G)$ then $\deg_G(x) = 2$.

Definition 1.2. Let G be a graph. $L(G) = L_G$ the *line graph* of G is defined where $V(L_G) = E(G)$ and $(e, f) \in E(L_G)$ if and only if e, f are incident to the same vertex in G.

Lemma 1.3. *For each $v \in V(G)$, if it is incident to $e_1, \ldots, e_d \in E(G)$, then we can identify it with $H_v \subseteq L_G$, where $V(H_v)$ is the induced subgraph of $\{e_1, \ldots, e_d\}$, H_v is isomorphic to K_d and $H_v \cap H_u \neq \varnothing$ if and only if $(u, v) \in E(G)$, in which case $H_v \cap H_u = \{(u, v)\}$.*

Proof. H_v is isomorphic to K_d since every $e, f \in V(H_v)$ are neighbours (they are all incident to v). The intersection property follows $V(H_v \cap H_u) = V(H_v) \cap V(H_u)$. □

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Lemma 1.4. *For $e = (v, u) \in V(L_G)$ (where $v, u \in V(G)$ were neighbours in G),* $\deg_{L_G}(e) = \deg_G(v) + \deg_G(u) - 2$.

Proof. e is only neighbours with $f \in E(G)$ where $f \neq e$ incident to v or to u, and there are $\deg_G(v) + \deg_G(u) - 2$ of those. □

Proposition 1.5. *Let $G \in \mathcal{F}_k$ be k-regular.*

$$\phi : \mathcal{F}_k \to \mathcal{F}_k, G \mapsto L(D(G))$$

is a mapping where every $v \in V(G)$ can be identified to $H_v \subseteq \phi(G)$ whose vertex set are edges incident to v in G. H_v is isomorphic to K_k and H_v is joined by an edge to H_u if and only if $(u, v) \in E(G)$.

Remark. ϕ indeed maps into $\mathcal{F}_k$ by 1.4, since every $e \in E(D_G)$ joins a vertex of degree k (from $V(G)$) and one of degree 2 (from $E(G) \subseteq V(D_G)$). So $e \in V(\phi(G))$ would have degree $k + 2 - 2 = k$. We call ϕ the *vertex replacement map.*

Proof. Since $V(G) \subseteq V(D(G))$ and the degrees stay unchanged, we use H_v from 1.3 to get that they are isomorphic to K_k and that they share a vertex with each of H_e, where $e \in E(G)$ was added by the subdivisions and e is incident to v and some u neighbour of v. But this means H_e also shares a vertex with H_u (and $H_u \cap H_v = \varnothing$ since $u, v \in V(G)$ can never be neighbours in D_G), which means H_v shares an edge with H_u.

By the degree of every $x \in V(H_v)$, since they have $k - 1$ degrees inside H_v, they must have precisely one degree out of H_v, hence the *only if.* □

2 The spectral properties of the vertex replacement map

In this section, we establish a relationship between $\operatorname{Spec} G$ and $\operatorname{Spec} \phi(G)$, via the characteristic polynomials of G and $\phi(G)$ (the characteristic polynomial of G is defined to be $\chi_G = \chi_{\mathbf{A}(G)}$, where $\mathbf{A}(G)$ is the adjacency matrix of G). The algebraic graph theory results of this section are due to Cvetković et al. [2].

We start with a representation of the adjacency matrix of D_G.

Lemma 2.1. *Let G have n vertices and ℓ edges. Let $\mathbf{X}_{n\times\ell} = \mathbf{X}(G)$ be the undirected incidence matrix of G ($\mathbf{X}_{ve} = 1$ if and only if e incident to v). Then we can write*

$$\mathbf{A}(D_G) = \begin{pmatrix} 0_{\ell\times\ell} & \mathbf{X}(G)^\intercal \\ \mathbf{X}(G) & 0_{n\times n} \end{pmatrix}.$$

Proof. The first ℓ rows and columns of $\mathbf{A}(D_G)$ represent $E(G)$ and the following n represent $V(G)$. $\mathbf{A}_{ij} = 0$ if $i, j \in V(G)$ or $i, j \in E(G)$ and $\mathbf{A}_{ij} = 1$ if and only if $\mathbf{X}(G)_{ij} = 1$ (if $i \in V(G)$ and $j \in E(G)$) or $\mathbf{X}(G)_{ji} = 1$ (if $i \in E(G)$ and $j \in V(G)$). □

Now we state and prove two well-known results in matrix theory.

Lemma 2.2. *If* $\mathbf{M}$ *is invertible and* $\mathbf{T}$ *is square,*

$$\det\begin{pmatrix}\mathbf{M} & \mathbf{X}\\ \mathbf{Y} & \mathbf{T}\end{pmatrix} = \det\mathbf{M}\det(\mathbf{T}-\mathbf{Y}\mathbf{M}^{-1}\mathbf{X}).$$

Proof. Note (assume $\mathbf{M}_{\ell\times\ell}$ and $\mathbf{T}_{n\times n}$)

$$\begin{pmatrix}\mathbf{M}^{-1} & 0_{\ell\times n}\\ 0_{n\times\ell} & \mathbf{I}_n\end{pmatrix}\begin{pmatrix}\mathbf{I}_\ell & -\mathbf{X}\\ 0_{n\times\ell} & \mathbf{I}_n\end{pmatrix} = \begin{pmatrix}\mathbf{M}^{-1} & -\mathbf{M}^{-1}\mathbf{X}\\ 0_{n\times\ell} & \mathbf{I}_n\end{pmatrix}$$

is precisely the product of column operations that eliminate $\mathbf{M}$ and $\mathbf{X}$. Then

$$\begin{pmatrix}\mathbf{M} & \mathbf{X}\\ \mathbf{Y} & \mathbf{T}\end{pmatrix}\begin{pmatrix}\mathbf{M}^{-1} & -\mathbf{M}^{-1}\mathbf{X}\\ 0_{n\times\ell} & \mathbf{I}_n\end{pmatrix} = \begin{pmatrix}\mathbf{I}_\ell & 0_{\ell\times n}\\ \mathbf{Y}\mathbf{M}^{-1} & \mathbf{T}-\mathbf{Y}\mathbf{M}^{-1}\mathbf{X}\end{pmatrix}$$
$$\implies \det\begin{pmatrix}\mathbf{M} & \mathbf{X}\\ \mathbf{Y} & \mathbf{T}\end{pmatrix}(\det\mathbf{M})^{-1} = \det(\mathbf{T}-\mathbf{Y}\mathbf{M}^{-1}\mathbf{X}).$$ □

Lemma 2.3. *If* $\mathbf{X}_{m\times n}$*,* $\mathbf{Y}_{n\times m}$ *are matrices and* $\chi_{\mathbf{A}}$ *denotes the characteristic polynomial of a square matrix* $\mathbf{A}$*, then*

$$\lambda^n\chi_{\mathbf{XY}}(\lambda) = \lambda^m\chi_{\mathbf{YX}}(\lambda).$$

Proof. We start by the case where $\mathbf{X},\mathbf{Y}$ are $n\times n$, and $\mathbf{X}$ is invertible. Then

$$\mathbf{X}(\mathbf{YX})\mathbf{X}^{-1} = \mathbf{XY} \implies \det(\lambda\mathbf{I}_n-\mathbf{XY}) = \det(\mathbf{X}(\lambda\mathbf{I}_n-\mathbf{YX})\mathbf{X}^{-1}) = \det(\lambda\mathbf{I}_n-\mathbf{YX}).$$

For arbitrary $\mathbf{X},\mathbf{Y}\in\mathbb{R}^{n^2}$, given fixed $\lambda\in\mathbb{R}$, if we consider

$$\psi_\lambda:\mathbb{R}^{n^2}\times\mathbb{R}^{n^2}\to\mathbb{R}, (\mathbf{X},\mathbf{Y})\mapsto\chi_{\mathbf{XY}}(\lambda);\ \ \phi_\lambda:\mathbb{R}^{n^2}\times\mathbb{R}^{n^2}\to\mathbb{R}, (\mathbf{X},\mathbf{Y})\mapsto\chi_{\mathbf{YX}}(\lambda)$$

then $\psi_\lambda=\phi_\lambda$ infinitely often (wherever one of $\mathbf{X},\mathbf{Y}$ is invertible), so since ψ_λ and ϕ_λ are polynomials, $\psi_\lambda=\phi_\lambda$ everywhere. Thus $\chi_{\mathbf{XY}}=\chi_{\mathbf{YX}}$ whenever $\mathbf{X},\mathbf{Y}$ are square.

Given $n>m$, we write

$$\mathbf{X}'_{n\times n} = \begin{pmatrix}\mathbf{X}_{m\times n}\\ 0_{(n-m)\times n}\end{pmatrix}, \mathbf{Y}'_{n\times n} = \begin{pmatrix}\mathbf{Y}_{n\times m} & 0_{n\times(n-m)}\end{pmatrix}.$$

Then

$$\mathbf{X}'\mathbf{Y}' = \begin{pmatrix}\mathbf{X}\\ 0_{(n-m)\times n}\end{pmatrix}\begin{pmatrix}\mathbf{Y} & 0_{n\times(n-m)}\end{pmatrix} = \begin{pmatrix}\mathbf{XY} & 0_{m\times(n-m)}\\ 0_{(n-m)\times m} & 0_{(n-m)\times(n-m)}\end{pmatrix};$$
$$\mathbf{Y}'\mathbf{X}' = \begin{pmatrix}\mathbf{Y} & 0_{n\times(n-m)}\end{pmatrix}\begin{pmatrix}\mathbf{X}\\ 0_{(n-m)\times n}\end{pmatrix} = \mathbf{YX}$$
$$\implies \chi_{\mathbf{X}'\mathbf{Y}'}(\lambda) = \det\begin{pmatrix}\lambda\mathbf{I}_m-\mathbf{XY} & 0_{m\times(n-m)}\\ 0_{(n-m)\times m} & \lambda\mathbf{I}_{n-m}\end{pmatrix} = \lambda^{n-m}\chi_{\mathbf{XY}}(\lambda) = \chi_{\mathbf{Y}'\mathbf{X}'}(\lambda) = \chi_{\mathbf{YX}}(\lambda).$$ □

We need one last lemma relating χ_G and $\chi_{L(G)}$.

Lemma 2.4. *Let G be a graph with n vertices, ℓ edges, degree matrix $\mathbf{D}$ and adjacency matrix $\mathbf{A}$. Then*

$$\chi_{L_G}(\lambda - 2) = \lambda^{\ell - n} \det(\lambda \mathbf{I}_n - \mathbf{A} - \mathbf{D}).$$

Proof. Note $\mathbf{A}(L_G) + 2\mathbf{I}_\ell = \mathbf{X}(G)^\intercal \mathbf{X}(G)$. Indeed for $i \neq j$, $\mathbf{A}(L_G)_{ij} = 1$ if and only if i and j are incident to the same vertex, which is precisely when the i-th and j-th columns of $\mathbf{X}(G)$ have a 1 at the same position. For $i = j$, the i-th column of $\mathbf{X}(G)$ always has precisely two 1s, so $(\mathbf{X}(G)^\intercal \mathbf{X}(G))_{ii} = 2$. This implies

$$\chi_{L_G}(\lambda - 2) = \det(\lambda \mathbf{I}_\ell - 2\mathbf{I}_\ell - \mathbf{A}(L_G)) = \chi_{\mathbf{X}(G)^\intercal \mathbf{X}(G)}(\lambda).$$

On the other hand $\mathbf{X}(G)\mathbf{X}(G)^\intercal = \mathbf{A} + \mathbf{D}$. Indeed for $i \neq j$, $\mathbf{A}_{ij} = 1$ if and only if i and j are incident to the same edge, which is when i is adjacent to j. If $i = j$ then $(\mathbf{X}(G)\mathbf{X}(G))_{ii} = \deg(i) = \mathbf{D}_{ii}$. The result then follows from 2.3:

$$\chi_{L_G}(\lambda - 2) = \chi_{\mathbf{X}(G)^\intercal \mathbf{X}(G)}(\lambda) = \lambda^{\ell-n} \chi_{\mathbf{X}(G)\mathbf{X}(G)^\intercal}(\lambda) = \lambda^{\ell-n}\chi_{\mathbf{A}+\mathbf{D}}(\lambda) = \lambda^{\ell-n} \det(\lambda \mathbf{I}_n - \mathbf{A} - \mathbf{D}). \qquad \square$$

Now we establish the main result between χ_G and $\chi_{\phi(G)}$.

Theorem 2.5. *Let $G \in \mathcal{F}_k$ have n vertices and $\ell = nk/2$ edges. Then*

$$\chi_{\phi(G)}(\lambda) = (\lambda(\lambda+2))^{\ell-n} \chi_G(\lambda^2 + (2-k)\lambda - k).$$

Proof. Recall

$$\mathbf{A}(D_G) = \begin{pmatrix} 0_{\ell\times\ell} & \mathbf{X}(G)^\intercal \\ \mathbf{X}(G) & 0_{n\times n} \end{pmatrix}, \ \mathbf{D}(D_G) = \begin{pmatrix} 2\mathbf{I}_\ell & 0_{\ell\times n} \\ 0_{n\times\ell} & k\mathbf{I}_n \end{pmatrix}.$$

By 2.4

$$\chi_{\phi(G)}(\lambda) = (\lambda+2)^{\ell-n} \det((\lambda+2)\mathbf{I}_{\ell+n} - \mathbf{A}(D_G) - \mathbf{D}(D_G)).$$

We note that if we write (using 2.1)

$$\mathbf{A}(D_G) = \begin{pmatrix} 0_{\ell\times\ell} & \mathbf{X}(G)^\intercal \\ \mathbf{X}(G) & 0_{n\times n} \end{pmatrix},$$

then (recall Section 1: $\deg_{D_G}(v) = 2$ if $v \in E(G)$, k if $v \in V(G)$)

$$\mathbf{D}(D_G) = \begin{pmatrix} 2\mathbf{I}_\ell & 0_{\ell\times n} \\ 0_{n\times\ell} & k\mathbf{I}_n \end{pmatrix}.$$

Then by 2.2

$$(\lambda+2)\mathbf{I}_{\ell+n} - \mathbf{A}(D_G) - \mathbf{D}(D_G) = \begin{pmatrix} (\lambda+2-2)\mathbf{I}_\ell & -\mathbf{X}(G)^\intercal \\ -\mathbf{X}(G) & (\lambda+2-k)\mathbf{I}_n \end{pmatrix} = \begin{pmatrix} \lambda\mathbf{I}_\ell & -\mathbf{X}(G)^\intercal \\ -\mathbf{X}(G) & (\lambda-k+2)\mathbf{I}_n \end{pmatrix}$$

$$\implies \det \begin{pmatrix} \lambda\mathbf{I}_\ell & -\mathbf{X}(G)^\intercal \\ -\mathbf{X}(G) & (\lambda-k+2)\mathbf{I}_n \end{pmatrix} = \lambda^\ell \det((\lambda-k+2)\mathbf{I}_n - \lambda^{-1}\mathbf{X}(G)\mathbf{X}(G)^\intercal)$$

$$= \lambda^{\ell-n} \det((\lambda(\lambda-k+2)-k)\mathbf{I}_n - \mathbf{A}(G)) = \lambda^{\ell-n}\chi_G(\lambda(\lambda-k+2)-k)$$

$$\implies \chi_{\phi(G)}(\lambda) = (\lambda(\lambda+2))^{\ell-n}\chi_G(\lambda(\lambda-k+2)-k). \qquad \square$$

Corollary 2.6. *Let $G \in \mathcal{F}_k$ have n vertices and $\ell = nk/2$ edges and let $f(\lambda) = \lambda^2 + (2-k)\lambda - k$. Then*

$$\operatorname{Spec}\phi(G) = \{0\}^{(\ell-n)} \cup \{-2\}^{(\ell-n)} \cup f^{-1}(\operatorname{Spec} G).$$

where $\operatorname{Spec} G$ *and* $\operatorname{Spec}\phi(G)$ *are understood to be multisets (and $\{x\}^{(\ell-n)}$ denotes the multiset with x having multiplicity $\ell - n$) .*

Proof. Everything follows 2.5: if $\lambda = 0$ or -2 then $\chi_{\phi(G)}(\lambda) = 0$; If $\lambda \in f^{-1}(\operatorname{Spec} G)$ then $f(\lambda) \in \operatorname{Spec} G$ so $\chi_G(f(\lambda)) = 0$ so $\chi_{\phi(G)}(\lambda) = 0$. The multiplicities follow from the multiplicity of λ as a root of $\chi_{\phi(G)}$. □

3 Spectrum of a sequence of vertex replacement graphs

Kollár and Sarnak [4] found a set A such that for $G \in \mathcal{F}_3$, if $\operatorname{Spec} G \subseteq A$ then $\operatorname{Spec}\phi(G) \subseteq A$. This means that $[-3,3] \setminus A$ is guaranteed to be *gapped* by the spectrum of $\{\phi^n(G) : n \geq 0\}$, if $\operatorname{Spec} G \subseteq A$. We will do the same for $G \in \mathcal{F}_k$.

We use the notation

$$f(\lambda) = f_k(\lambda) = \lambda^2 + (2-k)\lambda - k$$

and

$$\Gamma = \Gamma_k = \bigcap_{j=0}^{\infty} f_k^{-j}([-k,k]).$$

We are looking for a set $A = A_k \subseteq [-k,k]$ such that

$$\operatorname{Spec} G \subseteq A_k \implies \operatorname{Spec}\phi(G) = f_k^{-1}(\operatorname{Spec} G) \cup \{0,2\} \subseteq A_k. \tag{3.1}$$

Note that if $f_k^{-j}(A_k \cup \{0,-2\}) \subseteq A_k$, $\forall j \geq 0$ then A_k would satisfy (3.1). We will proceed to show that $A_k = \Gamma_k \cup \bigcup_{j=0}^{\infty} f_k^{-j}(\{0\})$ has the desired property.

Lemma 3.1. *Suppose $x \notin [-k,k]$. Then $f_k(x) \notin [-k,k]$. In consequence, $x \notin \Gamma_k$ and $f_k^j(x) \neq 0$, $\forall j \geq 0$.*

Proof. Assume $x > k$ i.e. $x = k + \epsilon$. Then

$$f_k(x) = (k+\epsilon)^2 + (2-k)(k+\epsilon) - k = (k+\epsilon)(\epsilon+2) - k > (k+\epsilon)(\epsilon+2) - k - \epsilon = (k+\epsilon)(\epsilon+1) > k.$$

Now assume $x < -k$ i.e. $x = -k - \epsilon$. Then (recall $k \geq 2$ so $2k - 3 \geq 1$)

$$f_k(x) = (-k-\epsilon)^2 + (k-2)(k+\epsilon) - k > (k+\epsilon)(k+\epsilon+k-2) - k - \epsilon = (k+\epsilon)(2k+\epsilon-3) > k. \quad \square$$

Theorem 3.2. *Fix $k \geq 2$. Let $A = \Gamma \cup \bigcup_{j=0}^{\infty} f^{-j}(\{0\})$. Then $f^{-j}(A \cup \{0,-2\}) \subseteq A \subseteq [-k,k]$, $\forall j \geq 0$. In other words, A satisfies (3.1)*

Proof. $\Gamma \subseteq [-k,k]$ and by 3.1 $\bigcup_{j=0}^{\infty} f_k^{-j}(\{0\}) \subseteq [-k,k]$ so $A \subseteq [-k,k]$. Note

$$f^{-j}(A \cup \{0,-2\}) = f^{-j}(\Gamma \cup \{-2\}) \cup \bigcup_{i=0}^{\infty} f^{-(i+j)}(\{0\}), \forall j \geq 0.$$

Clearly $\bigcup_{i=0}^{\infty} f^{-(i+j)}(\{0\}) \subseteq A$ by definition. Furthermore, $f(-2) = (-2)^2 + (2-k)(-2) - k = k$ and $f^2(-2) = f(k) = k^2 + (2-k)k - k = k$ so $f^j(-2) \in [-k, k]$ for all $j \geq 0$ so $-2 \in f^{-j}([-k,k]), \forall j \geq 0$ i.e. $-2 \in \Gamma$ so $\Gamma \cup \{-2\} = \Gamma$. Thus we will be done if we can show that $f^{-j}(\Gamma) \subseteq \Gamma, \forall j \geq 0$.

Via induction this is equivalent to saying that $f^{-1}(\Gamma) \subseteq \Gamma$. If $x \in f^{-1}(\Gamma_k)$ then $f(x) \in \Gamma$ so by definition

$$\forall j \geq 0 : f(x) \in f^{-j}([-k,k]) \implies x \in f^{-(j+1)}([-k,k])$$

Now we only need $x \in f^{-0}([-k,k]) = [-k,k]$, but this follows 3.1: $x \notin [-k,k] \implies f(x) \notin \Gamma$. Thus we have shown $x \in \Gamma$. □

What we showed is that suppose we are given $G \in \mathcal{F}_k$ such that $\operatorname{Spec} G \subseteq A_k$, then for the sequence $\{G_j = \phi^j(G) : j \geq 0\}$, $\operatorname{Spec} G_j \subseteq A_k, \forall j \geq 0$.

A natural question now is the following: which points are known to be in A_k?

Proposition 3.3. *Fix $k \geq 2$. Then $\bigcup_{j=0}^{\infty} f^{-j}(\{-1, 0, k\}) \subseteq A$.*

Proof. $\bigcup_{j=0}^{\infty} f^{-j}(\{0\}) \subseteq A$ is by definition. Note $f(-1) = 1 - 2 + k - k = -1$ and (recall) $f(k) = k$. Thus for any x, if there exists $j \geq 0$ such that $f^j(x) \in \{-1, k\}$, then $f^{i+j}(x) \in \{-1, k\}$ for all $i \geq 0$ and furthermore $f^i(x) \in [-k, k]$ for all $i = 0, \dots, j$ by 3.1. This means $x \in \Gamma \subseteq A$. □

We now show some examples where (3.1) can be applied via 3.3. The graphs used in the following examples were investigated in Biggs [1].

Example 3.4. We know (this can be shown by the fact that $\mathbf{A}(K_{k+1})$ is circulant)

$$\operatorname{Spec} K_{k+1} = \{k\}^{(1)} \cup \{-1\}^{(k)}.$$

So let $G = K_{k+1}$. Then $\{G_j = \phi^j(G) : j \geq 0\}$ has $\operatorname{Spec} G_j \subseteq A_k, \forall j \geq 0$.

Example 3.5. Define H_s to be the graph on $2s$ vertices obtained by removing s disjoint edges (i.e. a perfect matching) from K_{2s}. H_s is called a *hyperoctahedral graph* and H_3 is the octahedral graph. Note $H_s \in \mathcal{F}_{2(s-1)}$. Again via the fact that $\mathbf{A}(H_s)$ is circulant, we know

$$\operatorname{Spec} H_s = \{2(s-1)\}^{(1)}\{0\}^{(s)}\{-2\}^{(s-1)}.$$

Recall $-2 \in f_k^{-1}(\{k\})$ for any $k \geq 2$. Thus for $s \geq 2$, if we let $G = H_s$ then $\{G_j = \phi^j(G) : j \geq 0\}$ has $\operatorname{Spec} G_j \subseteq A_{2(s-1)}, \forall j \geq 0$.

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